INVESTIGATION OF A PECULIARITY OF NONLINEAR EQUATIONS OF CONTROL SYSTEMS IN THE CASE OF MULTIPLE ROOTS BY APPLICATION OF THE THEORY OF MATRICES

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The reduction of the equations of direct control to a special canonical form is studied, in which a nonlinear function of the control unit enters into the system with coefficients zero and one. If there exist prime characteristic roots of the matrix of the coefficients of the initial system the method of reduction coincides with the method given in book [1]. For the case of multiple characteristic roots of this matrix the order of the canonical system, equal to the power of the minimal polynomial of the original matrix, may be less than the order of the initial system. It will be shown that in the latter case, the stability of the original system does not always follow from the stability of the canonical system of equations, although instability of the original system always follows from the instability of the canonical system. A method is presented for constructing, from the solutions of the canonical system, the solutions of the original system and conversely, from the solutions of the original system, for constructing the solutions of the canonical system.

1. The components $A_\gamma^{(k)}$ of matrix A of dimension $(n \times n)$ may be determined as the numerator of the decomposition $(\lambda E_{n,n} - A)^{-1}$ into simple fractions [2]

$$(\lambda E_{n,\,n} - A)^{-1} = \frac{F(\lambda)}{\Delta(\lambda)} = \frac{D_{n-1}(\lambda)\, C(\lambda)}{D_{n-1}(\lambda)\, \Delta_r(\lambda)} = \sum_{k=1}^{s} \sum_{\gamma=1}^{m_k} \frac{A_\gamma^{(k)}}{(\lambda - \lambda_k)^\gamma} \tag{1.1}$$

$$\Delta(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \ldots (\lambda - \lambda_s)^{n_s} \qquad (n_1 + \ldots + n_s = n) \tag{1.2}$$

$$\Delta_r(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_s)^{m_s} \qquad (m_1 + \ldots + m_s = m \leqslant n) \tag{1.3}$$

Here $F(\lambda)$ and $C(\lambda)$ are adjoint and reduced adjoint [2] matrices for $(\lambda E_{n,n} - A)$; $D_{n-1}(\lambda)$ is the greatest divisor of all minors of $(n - 1)$th order of the matrix $(\lambda E_{n,n} - A)$; $\Delta(\lambda)$ and $\Delta_r(\lambda)$ are the characteristic and minimal polynomials [2] of matrix A [i.e. $\Delta_r(\lambda)$ is the major invariant multiplier of the characteristic matrix $(\lambda E_{n,n} - A)$]; s is the number of distinct characteristic roots of matrix A.

The components $A_\gamma^{(k)}$ turn out to be linearly independent and different from zero and are completely determined by the matrix A.

The components of the matrix A may be used for finding in practice the function of the matrix A.

$$f(A) = \sum_{k=1}^{s} \left[f(\lambda_k) A_1^{(k)} + f'(\lambda_k) A_2^{(k)} + \cdots + \frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} A_{m_k}^{(k)} \right] \tag{1.4}$$

In particular, for the function e^{At} we have the relation

$$e^{At} = \sum_{k=1}^{s} \left[A_1^{(k)} + A_2^{(k)} t + \cdots + A_{m_k}^{(k)} \frac{t^{m_k-1}}{(m_k-1)!} \right] e^{\lambda_k t} \tag{1.5}$$

In turn, functions of the matrix may be used for the integration of a system of linear differential equations with constant coefficients.

In the present note the components of a matrix are used for the investigation of a control system.

2. The use of components of a matrix for the transformation of the equations of the theory of control. We consider a system of differential equations of the first order

$$\dot{x}_k = \sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha + h_k \varphi(\sigma) \quad (k = 1, \ldots, n), \qquad \sigma = \sum_{\alpha=1}^{n} r_\alpha x_\alpha \tag{2.1}$$

which we may rewrite in the form

$$x = Bx + H\varphi(\sigma), \qquad \sigma = Rx \tag{2.2}$$

Here x and H are column matrices of the elements x_k and h_k, respectively, B is a square matrix of the coefficients $b_{k\alpha}$ of dimension $(n \times n)$, R is a row matrix of the elements r_α.

If in Equations (2.1) the quantity σ represents an arbitrary function of time, then the solution for x may be written in the form

$$x = e^{Bt} x_0 + \int_0^t e^{B(t-\tau)} H\varphi\{\sigma(\tau)\}\, d\tau \tag{2.3}$$

Analogous to (1.5) the function e^{Bt} has the form

$$e^{Bt} = \sum_{k=1}^{s} \left[B_1^{(k)} + B_2^{(k)} t + \cdots + B_{m_k}^{(k)} \frac{t^{m_k - 1}}{(m_k - 1)!} \right] e^{\lambda_k t} \tag{2.4}$$

The quantities $\lambda_1, \ldots, \lambda_s$ are distinct characteristic roots of matrix B

$$|\lambda E_{n,n} - B| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \ldots (\lambda - \lambda_s)^{n_s} \tag{2.5}$$

Consequently, relation (2.3) may be given the form

$$x = e^{Bt} x_0 + U\xi^\circ \tag{2.6}$$

where U is a rectangular $(n \times m)$ matrix, the columns of which are products of components of matrix B by column H

$$U = \| B_\gamma^{(k)} H \| = \| B_1^{(1)} H,\ B_2^{(1)} H,\ \ldots,\ B_{m_s}^{(s)} H \| \tag{2.7}$$

and ξ° is a complicated column matrix

$$\xi^\circ = \left\| \begin{matrix} \xi^{(1)\circ} \\ \cdots \\ \xi^{(s)\circ} \end{matrix} \right\| \tag{2.8}$$

the elements of which are the columns

$$\xi^{(k)\circ} = \left\| \begin{matrix} \xi_1^{(k)\circ} \\ \cdots \\ \xi_{m_k}^{(k)\circ} \end{matrix} \right\| = \int_0^t e^{\lambda_k (t-\tau)} \left\| \begin{matrix} 1 \\ \cdots \\ \dfrac{(t-\tau)^{m_k - 1}}{(m_k - 1)!} \end{matrix} \right\| \varphi\{\sigma(\tau)\}\, d\tau \quad (k=1, \ldots, s) \tag{2.9}$$

By immediate differentiation it is possible to convince oneself that ξ° is a particular solution of the system of equations

$$\begin{aligned} \dot{\xi}_1^{(k)} &= \lambda_k \xi_1^{(k)} + \varphi(\sigma) \\ \dot{\xi}_2^{(k)} &= \lambda_k \xi_2^{(k)} + \xi_1^{(k)} \qquad (k = 1, \ldots, s) \\ &\cdots\cdots\cdots\cdots \\ \dot{\xi}_{m_k}^{(k)} &= \lambda_k \xi_{m_k}^{(k)} + \xi_{m_k - 1}^{(k)} \end{aligned} \tag{2.10}$$

the matrix form of which is

$$\dot{\xi} = \Lambda\xi + G\varphi(\sigma) \tag{2.11}$$

where

$$\underset{(m\times m)}{\Lambda} = \left\| \begin{matrix} \Lambda^{(1)} & 0 & \ldots & 0 \\ 0 & \Lambda^{(2)} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & \Lambda^{(s)} \end{matrix} \right\|, \qquad \underset{(m\times 1)}{G} = \left\| \begin{matrix} G^{(1)} \\ G^{(2)} \\ \cdots \\ G^{(s)} \end{matrix} \right\|$$

$$\underset{(m_k\times m_k)}{\Lambda^{(k)}} = \left\| \begin{matrix} \lambda_k & 0 & \ldots & 0 \\ 1 & \lambda_k & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \lambda_k \end{matrix} \right\|, \qquad \underset{(m_k\times 1)}{G^{(k)}} = \left\| \begin{matrix} 1 \\ 0 \\ \ldots \\ 0 \end{matrix} \right\|$$

The general solution of Equations (2.11)

$$\xi = e^{\Lambda t}\xi_0 + \xi^\circ \tag{2.12}$$

depends on m arbitrary constants

$$\xi_{10}^{(k)}, \quad \xi_{20}^{(k)}, \ldots \xi_{m_k 0}^{(k)} \qquad (k = 1, \ldots, s)$$

Substituting ξ° from (2.12) into (2.6), we obtain

$$x = e^{Bt}x_0 - Ue^{\Lambda t}\xi_0 + U\xi \tag{2.13}$$

Equation (2.12) by itself represents the relation, with the aid of which for arbitrary value σ the solutions of Equations (2.1) are obtained from the solutions of Equations (2.10). The column x_0 represents by itself the column of initial values of the variables x_k, and the column ξ_0 is the column of arbitrary constants. In choosing the values of these constants we shall require that the product Rx be not dependent on x_0 and ξ_0. Then we have

$$Rx = RU\xi, \qquad Re^{Bt}x_0 = RUe^{\Lambda t}\xi_0 \tag{2.14}$$

If the product RU is denoted by Q,

$$Q = \| q_1^{(1)}, q_2^{(1)}, \ldots, q_{m_s}^{(s)} \|, \qquad q_\gamma^{(k)} = RB_\gamma^{(k)}H \tag{2.15}$$

then Equations (2.14) may be given in the form

$$Rx = Q\xi, \qquad Re^{Bt}x_0 = Qe^{\Lambda t}\xi_0 \tag{2.16}$$

In the last of these equations the expression for e^{Bt} takes the form (2.4). It is possible to show that the expression for $e^{\Lambda t}$ will have the analogous form

$$e^{\Lambda t} = \sum_{k=1}^{s} \left[\Lambda_1^{(k)} + \Lambda_2^{(k)}t + \ldots + \Lambda_{m_k}^{(k)} \frac{t^{m_k - 1}}{(m_k - 1)!}\right] e^{\lambda_k t} \tag{2.17}$$

and the number of components of $\Lambda_\gamma^{(k)}$ will equal the number of components of $B_\gamma^{(k)}$

Hence the second equation (2.16) is equivalent to the m equations

$$Q\,\Lambda_\gamma^{(k)}\xi_0 = RB_\gamma^{(k)}x_0 \qquad (k = 1, \ldots, s;\ \gamma = 1, \ldots, m_k) \tag{2.18}$$

with m unknowns $\xi_{10}^{(1)}, \xi_{20}^{(1)}, \ldots, \xi_{m_s 0}^{(s)}$. The rows of the determinant, composed of the coefficients of these unknowns, are the products of the rows of Q with the components $\Lambda_\gamma^{(k)}$. Because these components are linearly independent, this determinant is different from zero and the system (2.18) has a unique solution which we denote, with the aid of matrix V of dimension ($m \times n$), in the following form:

$$\xi_0 = Vx_0 \qquad \left(V = \| Q\,\Lambda_\gamma^{(k)}\|^{-1}\| RB_\gamma^{(k)}\|\right) \tag{2.19}$$

Here, for brevity in writing, the row matrices are written in the form of products of rows Q and R and the components of the corresponding matrices. Thus, if $\xi_0 = Vx_0$, then $Rx = Q\xi$ and the solution of Equations (2.2) with the aid of the relation

$$x = [e^{Bt} - Ue^{\Lambda t}V]\, x_0 + U\xi \tag{2.20}$$

may be obtained from the solution of the equations

$$\dot{\xi} = \Lambda\xi + G\varphi(\sigma), \qquad \sigma = Q\xi \tag{2.21}$$

the form of which by analogy with papers of Lur'e, Letov, Troitzkii may be called the canonical form of the equations of control systems. The difference of this form of the equations from the canonical form of the equations in the papers of the authors mentioned consists in that the order of the system of these equations coincides with the number of components of matrix B, and this number may be equal to the order of the original system (2.2) only in the case when the minimal polynomial coincides with the characteristic polynomial of matrix B. This circumstance occurs in particular when among the characteristic roots of matrix B none are multiple.

3. Properties of matrices U and V. Having differentiated the second equation of (2.16) with respect to time we obtain the equation

$$Re^{Bt}Bx_0 = Qe^{\Lambda t}\Lambda\xi_0 \tag{3.1}$$

which is analogous to Equation (2.16) and, consequently, $\Lambda\xi_0$ must satisfy the equation $\Lambda\xi_0 = VBx_0$. Substituting Vx_0 instead of ξ_0 into this equation we have

$$\Lambda Vx_0 = VBx_0 \quad \text{or} \quad \Lambda V = VB \tag{3.2}$$

since this must be fulfilled for an arbitrary column x_0. Also it is easily seen that

$$e^{\mathrm{B}t}\mathrm{H} = \mathrm{U}e^{\Lambda t}\mathrm{G} \tag{3.3}$$

Differentiating this equality with respect to time we obtain

$$\mathrm{B}e^{\mathrm{B}t}\mathrm{H} = \mathrm{U}\,\Lambda e^{\Lambda t}\mathrm{G} \quad \text{or} \quad (\mathrm{BU} - \mathrm{U}\,\Lambda)\,e^{\Lambda t}\mathrm{G} = 0 \tag{3.4}$$

the latter being based on (3.3).

Owing to the fact that $e^{\Lambda t}$G represents by itself a column of linearly independent functions

$$e^{\lambda_1 t}, \quad te^{\lambda_1 t}, \ldots, \quad t^{m_s - 1}e^{\lambda_s t} \tag{3.5}$$

the equality (3.4) is correct only when BU = UΛ. Assuming $t = 0$ in (3.3) we have H = UG.

By definition, Q = RU: multiplying (3.3) by R, we obtain the equation

$$\mathrm{R}e^{\mathrm{B}t}\mathrm{H} = \mathrm{Q}e^{\Lambda t}\mathrm{G} \tag{3.6}$$

analogous to Equation (2.16); consequently

$$\mathrm{G} = \mathrm{VH} \tag{3.7}$$

Replacing ξ_0 in the second equation of (2.16) by Vx_0, we have

$$\mathrm{R}e^{\mathrm{B}t}x_0 = \mathrm{Q}e^{\Lambda t}\mathrm{V}x_0 \tag{3.8}$$

Taking into account that (3.8) must be satisfied for arbitrary column x_0 and arbitrary time $t > 0$, we obtain

$$\mathrm{R}e^{\mathrm{B}t} = \mathrm{Q}e^{\Lambda t}\mathrm{V}, \qquad \mathrm{R} = \mathrm{QV} \tag{3.9}$$

Multiplying the equality (3.3) on the left by the matrix V we have

$$\mathrm{V}e^{\mathrm{B}t}\mathrm{H} = \mathrm{VU}e^{\Lambda t}\mathrm{G}$$

Since from the equality (3.2) follows $e^{\Lambda t}\mathrm{V} = \mathrm{V}e^{\mathrm{B}t}$ then $e^{\Lambda t}\mathrm{VH} = \mathrm{VU}e^{\Lambda t}\mathrm{G}$ or $e^{\Lambda t}\mathrm{G} = \mathrm{VU}e^{\Lambda t}\mathrm{G}$, which is possible only when

$$\mathrm{VU} = \mathrm{E}_{m,\,m} \tag{3.10}$$

Finally, if we multiply U on the right by V and square this product we then obtain

$$(\mathrm{UV})^2 = \mathrm{UVUV} = \mathrm{UE}_{m,\,m}\mathrm{V} = \mathrm{UV} \tag{3.11}$$

Hence it follows that the product UV is an idempotent matrix [2] for which the relations

$$\mathrm{BUV} = \mathrm{U}\,\Lambda\mathrm{V} = \mathrm{UVB} \tag{3.12}$$

hold.

Summing up what has been said, we have

$$
\begin{array}{ccc}
e^{Bt}H = Ue^{\Lambda t}G, & BU = U\Lambda, & UG = H \\
Re^{Bt} = Qe^{\Lambda t}V, & \Lambda V = VB, & QV = R \\
Q = RU, & VU = E_{m,m}, & VH = G \\
(UV)^2 = UV, & RUV = R, & UVH = H
\end{array}
$$

4. Certain theorems concerning control systems with multiple roots. We rewrite the systems (2.2) and (2.21) in the form

$$\dot{x} = Bx + H\varphi(Rx) \tag{4.1}$$

$$\dot{\xi} = \Lambda\xi + G\varphi(Q\xi) \tag{4.2}$$

According to what has been said above, in order to obtain a solution of Equation (4.1) with initial condition $x(0) = x_0$, it is possible first to find a solution for ξ in Equations (4.2) with initial condition $\xi(0) = Vx_0$ and then, in correspondence with Formulas (2.20) and (3.2), to construct the solution of Equations (4.1)

$$x = (E_{n,n} - UV)e^{Bt}x_0 + U\xi \tag{4.3}$$

If it is necessary to construct a solution for ξ in Equation (4.2) with initial condition $\xi(0) = \xi_0$, then it is possible, choosing the column x_0 to satisfy the relation $\xi_0 = Vx_0$, to have a solution for x in Equation (4.1) with initial condition $x(0) = x_0$ and then by means of the formula $\xi = Vx$ to construct a solution of Equation (4.2). In fact, if $\xi = Vx$ is substituted into Equation (4.2)

$$V\dot{x} - \Lambda Vx - G\varphi(QVx) = 0 \tag{4.4}$$

and account is taken of Equations (3.2), (3.7), (3.9) then it is possible to construct the identity

$$V[\dot{x} - Bx - H\varphi(Rx)] = 0 \tag{4.5}$$

since x is a solution of Equation (4.1).

We pass in Equation (4.3) to the analysis of the expression in parentheses. If the characteristic polynomial of matrix B coincides with the minimal polynomial of matrix B, then $m = n$ and

$$E_{n,n} - UV = 0$$

This follows from the fact that U and V will in this case be $(n \times n)$ matrices and by virtue of the equality (3.10)

$$VU = E_{n,n} = UV$$

But if the degree of the minimal polynomial is less than the degree of the characteristic polynomial ($m < n$), which is possible only for the case of multiple characteristic roots of the matrix B, then

$$E_{n,n} - UV \neq 0$$

since in this inequality the matrix $E_{n,n}$ has rank n, but the rank of the product UV by virtue of Sylvester's theorem would be equal to $m < n$.

We consider in detail the second case and prove for it the following theorems:

Theorem 1. If the solution of Equation (4.2) for ξ is unstable, then the solution of Equation (4.1) for x is also unstable.

Theorem 2. If the solution of Equation (4.2) for ξ is stable and if the multiplicity of the roots with positive real parts in the minimal polynomial of matrix B is equal to the multiplicity of the roots with positive real parts in the characteristic polynomial of matrix B, then the solution of Equation (4.1) for x is stable, but if smaller, then unstable.

The proof of the first theorem follows from the fact that the solution for ξ is obtained from the solution for x with the aid of the linear transformation $\xi = Vx$. Whence it is seen that in order for the solution for ξ to be unstable it is necessary for the solution for x to be also unstable. It is evident that the first theorem will be correct also for $m = n$.

For the proof of the second theorem we shall consider that matrix B in the initial solution has Jordan normal form (with the aid of a linear nonsingular transformation of the unknowns it is always possible to reduce it to this form). Each such matrix can always be decomposed into a number of blocks, equal in number to the distinct roots of the characteristic polynomial

$$B = \begin{Vmatrix} B^{(1)} & \ldots & 0 \\ \cdot & \cdot\cdot\cdot & \cdot \\ 0 & \ldots & B^{(s)} \end{Vmatrix}$$

where the dimension of a block is equal to the multiplicity of the root in the characteristic polynomial. As far as the matrix Λ is concerned, it already has such a block form where the dimension of the block $\Lambda^{(k)}$ in the matrix Λ is equal to the multiplicity of the root in the minimal polynomial. We decompose the matrices R, H and x into blocks so that the products RB, BH, Bx have meaning, and matrices Q, G and ξ by their structure already have this form.

Then it is possible to show that the matrices U, V and UV have quasi-diagonal form

$$U = \left\| \begin{matrix} U^{(1)} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & U^{(s)} \end{matrix} \right\|, \qquad V = \left\| \begin{matrix} V^{(1)} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & V^{(s)} \end{matrix} \right\|, \qquad UV = \left\| \begin{matrix} U^{(1)}V^{(1)} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & U^{(s)}V^{(s)} \end{matrix} \right\|$$

Here the blocks $U^{(k)}$ have dimension $(n_k \times m_k)$, and the blocks $V^{(k)}$ have dimension $(m_k \times n_k)$; the product UV is formed by the multiplication rule of block matrices. Here the rank of the block $U^{(k)}V^{(k)}$ is equal to m_k. Since

$$(E_{n,n} - UV)\,e^{Bt} = \left\| \begin{matrix} (E_{n_1,n_1} - U^{(1)}V^{(1)})\,e^{B^{(1)}t} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & (E_{n_s,n_s} - U^{(s)}V^{(s)})\,e^{B^{(s)}t} \end{matrix} \right\|$$

then having decomposed x_0 also into block form, we obtain

$$(E_{n,n} - UV)\,e^{Bt}x_0 = \left\| \begin{matrix} (E_{n_1,n_1} - U^{(1)}V^{(1)})\,e^{B^{(1)}t}x_0^{(1)} \\ \ldots \\ (E_{n_s,n_s} - U^{(s)}V^{(s)})\,e^{B^{(s)}t}x_0^{(s)} \end{matrix} \right\|$$

We consider the kth block

$$(E_{n_k,n_k} - U^{(k)}V^{(k)})\,e^{B^{(k)}t}x_0^{(k)}$$

For arbitrary $x_0^{(k)}$ this block may equal zero only if

$$E_{n_k,n_k} = U^{(k)}V^{(k)}$$

which is possible only for the case when the multiplicity of the root of the minimal polynomial coincides with the multiplicity of the root in the characteristic polynomial. But if this condition is not fulfilled then such an equality is not possible. Since the function $\exp \lambda_k t$ is a scalar multiplier of this block-column, then for $\operatorname{Re} \lambda_k > 0$ the magnitude of the elements of the block-column increase without bound as $t \to \infty$ and this also is evidence of instability, which was to be proved.

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